

Schwinger pair production at nonzero temperatures or in compact directions

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Abstract

Electric fields may decay by quantum tunneling: as calculated by Schwinger, an electron-positron pair may be summoned from the vacuum. In this paper I calculate the pair-production rate at nonzero temperatures. I find that at high temperatures the decay rate is dominated by a new instanton that involves both thermal fluctuation and quantum tunneling; this decay is exponentially faster than the rate in the literature. I also calculate the decay rate when the electric field wraps a compact circle (at zero temperature). The same new instanton also governs this rate: I find that for small circles decay is dominated by a process that drops the electric field by one unit, but does not produce charged particles.

A uniform electric field is classically stable but quantum mechanically unstable. In the semiclassical regime, the dominant decay channel is the nucleation of an electron-positron pair, which discharges a single unit of flux. Heisenberg & Euler [1] and then Schwinger [2] calculated the exponential dependency of the tunneling rate to be

$$\text{decay rate}_{T=L^{-1}=0} \sim \exp \left[-\frac{1}{\hbar} \frac{\pi m^2}{e |\vec{E}|} \right]. \quad (1)$$

Here m is the positron mass, e is the positron charge, and $|\vec{E}|$ is the electric field strength.

Barriers that may be traversed by quantum tunneling may also be traversed by thermal fluctuation. I will show that at high temperature ($T > T_c \equiv \hbar \frac{e|\vec{E}|}{2m}$) the electric field decays by a process in which the electron-positron pair first thermally fluctuates partway up the barrier to nucleation, and only then quantum tunnels through the rest. For $T > T_c$ this thermally-assisted rate is exponentially faster than the Schwinger process:

$$\text{decay rate}_{T>T_c} \sim \exp \left[-\frac{1}{\hbar} \frac{2m^2}{e |\vec{E}|} \arcsin \left[\frac{T_c}{T} \right] - \frac{m}{T} \sqrt{1 - \frac{T_c^2}{T^2}} \right]. \quad (2)$$

I will also consider the decay of an electric field that points down a compact direction of circumference L . I will show that for small circles ($L < L_c \equiv \frac{2m}{e|\vec{E}|}$) a real electron-positron pair is not produced; instead, the energy from discharging the flux is dumped into photons. For $L < L_c$ this is exponentially faster than the Schwinger process:

$$\text{decay rate}_{L<L_c} \sim \exp \left[-\frac{1}{\hbar} \frac{2m^2}{e |\vec{E}|} \arcsin \left[\frac{L}{L_c} \right] - \frac{mL}{\hbar} \sqrt{1 - \frac{L^2}{L_c^2}} \right]. \quad (3)$$

Throughout I will work in the semiclassical (and analogous ‘semicold’) approximation. This is justified if the Compton wavelength $\hbar m^{-1}$ of the electron is short compared to the other scales in the problem: the electron-positron separation at nucleation $\frac{2m}{e|\vec{E}|}$, the thermal wavelength $\hbar T^{-1}$, and the circumference of the circle L . In the semiclassical regime decay is slow and exponentially dominated by the tunneling exponent; I will not attempt to calculate prefactors.

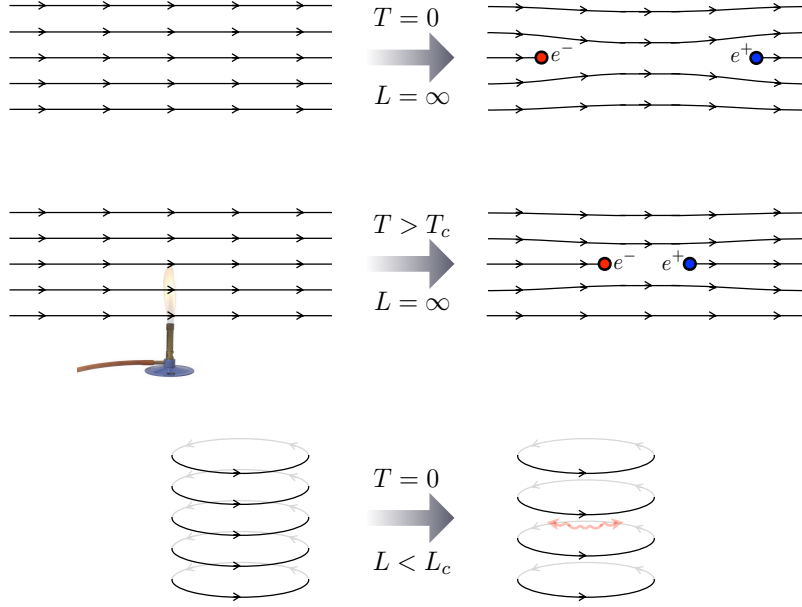


Figure 1: The three processes considered in this paper. **Top:** standard zero-temperature Schwinger pair production. An electric field line is snipped by the nucleation of an e^-e^+ pair. **Middle:** $T > T_c$ Schwinger pair production has the pair nucleated closer together, using energy that has been extracted from the heat bath. **Bottom:** When the electric field wraps a circle of small circumference $L < L_c$, decay drops the flux by one unit without producing a real pair; instead the energy is dumped into photons.

1. Schwinger pair production for $\mathbf{T} = \mathbf{L}^{-1} = \mathbf{0}$. A uniform electric field can release its energy by nucleating electron-positron pairs. But there is a barrier. To summon the pair from the vacuum has an upfront cost of $2m$, and this can only be repaid once the pair are far enough apart, $\Delta x > 2\bar{x}_0$ with

$$\bar{x}_0 = \frac{m}{e|\vec{E}|}. \quad (4)$$

To traverse this barrier, the pair must tunnel.

This tunneling can be thought of as the decay of the false vacuum of a 1+1-dimensional quantum field¹. In this language, pair production is the nucleation of a bubble of true vacuum [4–6]. The interior of the bubble has reduced electric field, and hence a lower energy density, $\Delta(\frac{1}{2}|\vec{E}|^2) = -e|\vec{E}|$; the surface of the bubble is the charged particle, which forms a thin wall of thickness $\hbar m^{-1}$ and tension m . Thus for Schwinger pair production, the semiclassical and thin-wall regimes coincide.

Since the electric field is uniform, the perpendicular spatial directions are passive spectators that can be integrated out. Consequently, we can use the results of 1+1-dimensional field theory, including Eqs. 1-3, in any number of spatial dimensions. The only exception to this

¹There are a number of different formalisms that can be used to perform essentially the same mathematical maneuvers [3]. In the appendix I will offer the alternative perspective of a direct WKB calculation of relativistic particle tunneling in the potential of Fig. 2.

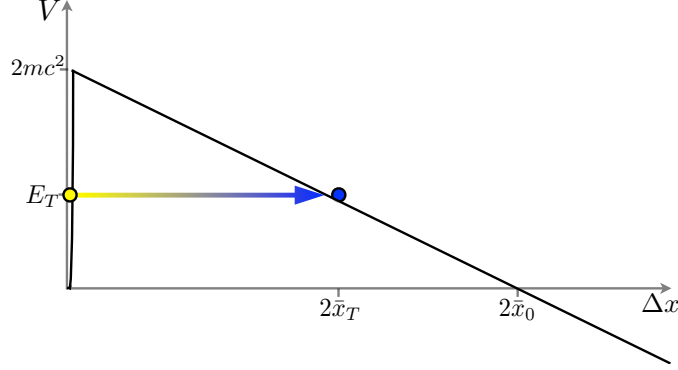


Figure 2: Traversing the barrier to nucleation. At zero temperature, the pair start colocated and then quantum tunnel to $\Delta x = 2\bar{x}_0$. For $T > T_c$, the pair first thermally fluctuate to an energy E_T , then quantum tunnel to $2\bar{x}_T$. In the appendix, the decay rate is calculated by directly applying the WKB formula to this potential.

universality is the $O(e^2)$ correction from the bilateral electromagnetic interaction of the electron and positron with each other: this is easily included, but it's dimension dependent and small for small e , and so will be neglected.

The Euclidean action for a bubble of true vacuum with surface at $x(\tau)$ is

$$I_{\text{Euclidean}} = m \times \text{perimeter} - e|\vec{E}| \times \text{area} = \int d\tau \left(m\sqrt{1 + \dot{x}^2} - e|\vec{E}|x \right). \quad (5)$$

(Not coincidentally, this is also the action of a relativistic particle in a uniform electric field.) The tunneling instanton is a saddle point of the Euclidean action with a single negative mode. The Euler-Lagrange equation tells us that for these boundary conditions, the instanton is a circular bubble of radius \bar{x}_0 (which w.l.o.g. we may center at the origin)

$$x^2 + \tau^2 = \bar{x}_0^2. \quad (6)$$

The exponential of the instanton action $\exp[-I_{\text{Euclidean}}/\hbar] = \exp[-(m \times 2\pi\bar{x}_0 - e|\vec{E}| \times \pi\bar{x}_0^2)/\hbar]$ gives the decay rate, Eq. 1. The classical Lorentzian evolution after nucleation is given by analytically continuing $\tau \rightarrow it$:

$$x^2 - t^2 = \bar{x}_0^2. \quad (7)$$

The electron and positron are nucleated at rest at $x = \mp\bar{x}_0$, and then accelerate apart.

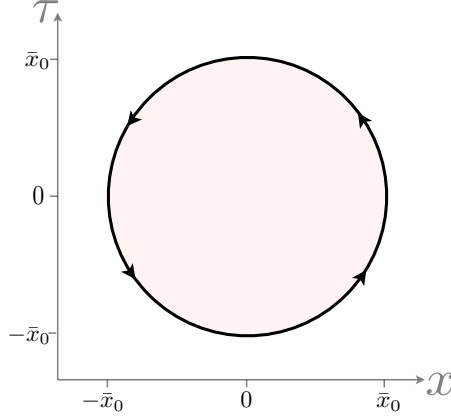


Figure 3: The Schwinger pair-production instanton is a bubble of radius $\bar{x}_0 = m/e|\vec{E}|$. Inside the bubble (shaded) the electric field is reduced and so the energy density is lower. The slice through $\tau = 0$ gives the $t = 0$ state immediately following nucleation: a momentarily stationary electron and positron separated by $2\bar{x}_0$.

2. Decay for $T > T_c$. At nonzero temperature, a particle may thermally fluctuate partway up the barrier before tunneling through the rest. The nonzero temperature makes Euclidean time compact [7, 8] with period $\beta \equiv \frac{\hbar}{T}$,

$$I_{\text{Euclidean}} = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \left(m\sqrt{1 + \dot{x}^2} - e|\vec{E}|x \right). \quad (8)$$

For $T < T_c$ (equivalently $\beta > \beta_c \equiv 2\bar{x}_0$) the complete Schwinger bubble still fits into the compact direction and the temperature has no effect on the tunneling exponent.

For $T > T_c$ the circular bubble no longer fits. Instead, the dominant instanton is a lens-shaped bubble bounded by two less-than-semicircular arcs, as shown in Fig. 4. Since the equations of motion are locally the same as for $T = 0$, the bubble walls must still be segments of a circle of radius \bar{x}_0 . The bubble has maximum width at $\tau = 0$ where $x = \pm\bar{x}_T$ with

$$\bar{x}_T = \bar{x}_0 \left(1 - \sqrt{1 - \frac{T_c^2}{T^2}} \right), \quad (9)$$

and narrows to a vertex at $x = 0, \tau = \pm\beta/2$. The exponential of the instanton action $\exp[-I_{\text{Euclidean}}/\hbar]$ gives the decay rate, Eq. 2.

The classical Lorentzian evolution after nucleation is given by analytically continuing $\tau \rightarrow it$. The electron and positron are initially at rest at $x = \mp\bar{x}_T$, and then accelerate apart.

A stationary electron-positron pair separated by $2\bar{x}_T$ has energy

$$E_T = 2m\sqrt{1 - \frac{T_c^2}{T^2}}. \quad (10)$$

This energy has been extracted from the heat bath. In the appendix we will see that the

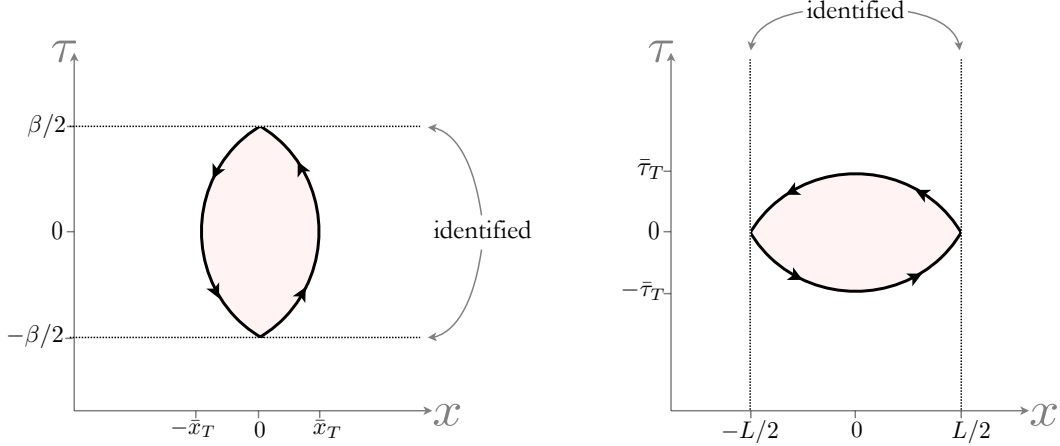


Figure 4: **Left:** for the nonzero-temperature instanton, Euclidean time is compact. Two less-than-semicircular arcs, each of radius of curvature \bar{x}_0 , meet at $x = 0, \tau = \pm\beta/2$. Inside the bubble (shaded) the electric field is reduced and so the energy density is lower. The slice through $\tau = 0$ gives the $t = 0$ state immediately following nucleation: a momentarily stationary electron and positron separated by $2\bar{x}_T$. **Right:** the same instanton does double duty, also describing the decay of a zero-temperature electric field pointing down a compact spatial direction of circumference L . Instead of producing an electron-positron pair, tunneling instead dumps the discharged energy into photons.

instanton has automatically calculated the optimal amount of energy to extract to make decay as fast as possible.

At high temperature the decay rate is

$$\text{rate}_{T \gg T_c} = \exp \left[-\frac{2m}{T} \left(1 - \frac{1}{6} \frac{T_c^2}{T^2} + \dots \right) \right] \quad (11)$$

The leading term in the exponent is the Boltzmann suppression of making a pair.

Let's compare thermally-assisted quantum tunneling in the potential of Fig. 2 to thermally-assisted quantum tunneling in two better-studied examples [7–13]; we'll find differences caused by the non-analyticity of the potential at $\Delta x = 0$. The first well-studied example is either quantum mechanics or quantum field theory when the barrier to be traversed is smooth. In that case, even at arbitrarily low nonzero temperatures the dominant process does not tunnel the whole way but instead receives at least a small thermal assist from the heat bath ($E_T > 0$); and furthermore above a critical temperature the process is purely thermal, meaning it fluctuates straight to the top of the barrier and doesn't quantum tunnel at all, so the instanton is uniform in τ [14, 15]. The second well-studied example is tunneling of higher-dimensional quantum fields whose barriers have been rendered non-analytic by taking the thin-wall limit. In this case there is a first-order transition in the decay rate: at a critical temperature, dominance jumps from purely quantum (spherical) instantons to purely thermal (cylindrical) instantons. Our 1+1-dimensional thin-wall pair-production bubbles are different from both of these examples. Instead, at low temperature ($T < T_c$) the dominant instanton is purely quantum (as in the thin-wall higher-dimensional case), but there is then a higher-order transition in the decay

rate, with the instanton acquiring an at-first-small thermal assist; furthermore, no matter how high the temperature the process never becomes purely thermal (and in particular the purely thermal solution with two conjoined thin walls has extra negative modes, analogous to [16], and never dominates tunneling).

The non-analyticity of the tunneling exponent at $T = T_c$ is an artifact of having taken the thin-wall limit, and $O(\hbar m^{-1})$ corrections will smooth it to a finitely sharp crossover.

Schwinger pair production at finite temperature has been considered before [17–25], but neither the instanton of Fig. 4 nor the rate of Eq. 2 has been derived².

3. Decay for $L < L_c$. Now consider an electric field at zero temperature that points down a compact direction of circumference L .

For $L > L_c \equiv 2\bar{x}_0$ the field decays by standard Schwinger pair production as described in Sec. 1. Since the space is compact, the electron and positron may then either collide with each other and annihilate into photons, or may miss each other and go round again, discharging a unit of flux on each lap [26].

For $L < L_c \equiv 2\bar{x}_0$ the Schwinger formula Eq. 4 predicts that the electron-positron pair would be created at a separation greater than L . This means the energy available from unwrapping a single unit of flux from the compact direction

$$\text{Energy Released} = e|\vec{E}|L < 2m \quad (12)$$

is insufficient to pay the $2m$ required to fabricate a pair. Instead the dominant process is a virtual pair circumnavigating the circle once and then annihilating into photons; the flux quantum jumps down by one unit without ever creating a real electron-positron pair.

The instanton that describes this process is shown in Fig. 4; the lens-shaped bubble extends as far as $\tau = \pm\bar{\tau}_T \equiv \pm\bar{x}_0(1 - \sqrt{1 - L^2/L_c^2})$. The finite- L instanton is thus the same mathematical solution as the finite- β instanton, only now with the two Euclidean dimensions swapped. Since the value of the Euclidean action is insensitive to how we label the axes, the rate Eq. 3 follows directly by substituting $\beta \rightarrow L$ into the thermal rate Eq. 2. For small circles the decay rate is

$$\text{rate}_{L \ll L_c} = \exp \left[-\frac{2mL}{\hbar} \left(1 - \frac{1}{6} \frac{L^2}{L_c^2} + \dots \right) \right]. \quad (13)$$

(A similar instanton may be used to calculate the discharge rate of an electric field between two capacitive plates, a problem considered without the use of instantons in [27].)

4. Discussion. In this paper, we have explored two complications we can add to the standard Schwinger story about tunneling in a uniform electric field: we have added a temperature, and

²For example, according to the result advanced in [25], as the temperature goes up the pair-production rate goes down. My rate Eq. 2 has the opposite behaviour.

we have compactified the direction down which the electric field points. In both cases, the rate is exponentially faster than that derived by Schwinger.

Associated with the faster process is some new phenomenology. In the thermal case, we found that the total energy of the electric field plus particles is not conserved. Energy is taken from the heat bath during the nucleation process that is never returned. In the compact case, the deviation is more dramatic. Unlike in the Schwinger process, no real pair is created. Instead a virtual pair mediates the unwrapping of a single unit of flux, but the pair annihilates before the tunneling process is complete, dumping the liberated energy into photons. In this process, not only is the generated current quantized in space, it is also quantized in time.

Equations 2 and 3 give the exponential contribution to the decay rate. In the semiclassical regime this is exponentially the most important contribution. A next step would be to calculate the leading contribution to the prefactor, which can be done by calculating the determinant of the matrix of small perturbations around the instanton [28].

This determinant is negative because, as befits a tunneling instanton, one of the eigenvalues is negative. Eigenmodes that leave the vertex unchanged are all positive [25], but the eigenmode that “reconnects” the vertex is negative. This mode deforms the sharp corner at which the two walls meet, smoothing it into an avoided crossing with lower Euclidean action.

If both $T > T_c$ and $L < L_c$ then both Euclidean directions are compact. For $\beta < L$ the dominant process is thermal nucleation, as in Sec. 2. For $\beta > L$ the dominant process is a quantum jump in the flux without producing charged particles, as in Sec. 3. There is also a process in which the quantum jump receives a thermal assist (governed by a Euclidean solution with four conjoined less-than-quarter-circular arcs) but it can be shown that this is subdominant. Giving the heat bath momentum in the compact direction would induce a chemical potential and make the Euclidean space in which the instanton lives deform from a rectangle to a parallelogram.

The two rates Eqs. 2 and 3 are related by the duality $\beta \leftrightarrow L$. Mathematically, this is because the same instanton controls both decays, only with relabelled axes. Since this relates large and small values of L/β , this is a high-temperature/low-temperature duality.

Experimentalists have sought to detect Schwinger pair production and measure the decay rate, Eq. 1, using graphene [31,32]. Perhaps Eq. 3 could be tested with carbon nanotubes.

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A. WKB calculation of tunneling rate. In the Sec. 2, we calculated the thermal decay rate using an instanton. In this appendix I will provide a potentially-enlightening alternative

perspective by directly applying the WKB formula to a particle tunneling through the barrier of Fig. 2. If the center of nucleation is $x = 0$, the positron moves in the potential

$$V_{e^+}[x] = \begin{cases} 0 & \text{for } x = 0 \\ m - e|\vec{E}|x & \text{for } x > 0, \end{cases} \quad (14)$$

and the electron moves towards negative x is a similar potential. To make the positron costs m , but the farther it goes in x , the more of that energy it recovers from the electric field.

The WKB tunneling suppression for a relativistic particle with no initial energy is $\exp[-I_{e^+}/\hbar]$ where [29]

$$I_{e^+} = 2 \int_0^{\bar{x}_0} dx \sqrt{V(x)[2m - V(x)]}. \quad (15)$$

(This reduces to $2 \int \sqrt{2mV}$ in the non-relativistic limit $V \ll m$; note that for $V > m$ higher barriers mean faster tunneling [30].) Applying Eq. 15 to Eq. 14 and including both I_{e^+} and I_{e^-} recovers the Schwinger rate Eq. 1.

Now let's turn on a temperature. A positron endowed by the heat bath with an energy E_i need only tunnel out to

$$\bar{x}_{E_i} = \frac{m - E_i}{e|\vec{E}|}. \quad (16)$$

The optimal value of E_i is determined by the interplay of a thermal (Boltzmann) suppression that wants E_i to be small, and a quantum (WKB) suppression that wants E_i to be big:

$$\text{rate}_{e^+}(E_i, T) = \exp \left[-\frac{E_i}{T} - \frac{2}{\hbar} \int_0^{\bar{x}_{E_i}} dx \sqrt{[V - E_i][2m - (V - E_i)]} \right] \quad (17)$$

$$= \exp \left[-\frac{E_i}{T} - \frac{2}{\hbar} \left(\frac{m^2}{2e|\vec{E}|} \arccos \left[\frac{E_i}{m} \right] - \frac{E_i \sqrt{m^2 - E_i^2}}{2e|\vec{E}|} \right) \right]. \quad (18)$$

For $T < T_c$ the decay rate is maximized at $E_i = 0$ and we recover the Schwinger result. For $T > T_c$ the optimal tradeoff between the quantum and thermal factors is given by $\partial_{E_i} \text{rate}_{e^+}(E_i, T) = 0$ as

$$2E_i \Big|_{\text{fastest}} = E_T = 2m \sqrt{1 - \frac{T_c^2}{T^2}}. \quad (19)$$

We see that the instanton automatically calculates the optimal value of E_i [7]. Applying Eq. 19 to Eq. 16 successfully recovers the separation \bar{x}_T of Eq. 9; and applying it to Eq. 18 successfully recovers the decay rate of Eq. 2.

This analysis also allows us to apportion the total decay suppression between the Boltzmann

factor and the WKB factor as $I_{\text{total}} = I_{\text{thermal}} + I_{\text{quantum}}$ where

$$I_{\text{thermal}} = \frac{2E_i}{T} = \frac{2m}{T} \sqrt{1 - \frac{T_c^2}{T^2}} \quad (20)$$

$$I_{\text{quantum}} = \frac{2m^2}{e|\vec{E}|} \arcsin\left[\frac{T_c}{T}\right] - \frac{m}{T} \sqrt{1 - \frac{T_c^2}{T^2}}. \quad (21)$$

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